Arbitrage

Binomial trees are a no arbitrage model.

S_0 = \$20 Call option strike = \$21 R_f = 12%
Stock price in 3 months will be either \$22 or \$18

A Generalized Approach

S_0 = stock price at time 0
f_0 = option price at time 0
u = size of up move
d = size of down move
Option can move up or down to S_0u or down to S_0d ($u > 1$; $d < 1$)
Proportional increase: $u - 1$
Proportional decrease: $1 - d$
Payoff in an up move = f_u
Payoff in a down move = f_d
Δ shares long in stock
Short 1 option

Previous Example:

$u = 1.1$	$d = 0.9$	$t = .25 = 3/12$
$R_f = 12\%$	$f_u = \$1$	$f_d = \$0$

Risk Neutral Valuation

Risk neutral – Individuals are indifferent to risk, they require no compensation for risk, the only thing that matters is the expected return. In a risk neutral world, the return on all securities is the risk-free rate.

What is the expected stock price at time T?

Important option principle: It is valid to assume the world is risk neutral when valuing options. The option price in the "real world" is the same as the option price in a risk neutral world.

Our example revisited

S_o = $20 S_T = $22 or $18
3 months step R_f = 12%

Two Step Binomial Tree
Call Option

S_0 = \$20	up or down 10% in each step	
R_f = 12%	X = \$21	step = 3 months

u = 1.1 d = .9 T = .25 p = .6523

European Put Option

S_0 = \$50	up or down 20% in each step	Expires in 2 years
R_f = 5%	X = \$52	step = 1 year

American Put Option – Notice that the inputs are the same

S_0 = \$50	up or down 20% in each step	Expires in 2 years
R_f = 5%	X = \$52	step = 1 year
u = 1.2	d = 0.8	

Risk neutral probability = p = $\frac{e^{.05(1)} - .8}{1.2 - .8}$ = .6282

Delta

u, d, Volatility and the "Real World"

μ = Expected return of stock
σ = Standard deviation of stock return
p^* = Probability of an up move in the real world

$$p^* = \frac{e^{\mu\Delta t} - d}{u-d}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u$$

American Call

μ =	12%	X =	\$50
S_0 =	\$55.00	R_f =	6%
σ =	40%		

Expires in 2 months, 1 month steps

$$u = e^{\sigma\sqrt{\Delta t}} = e^{.40(1/12)} = 1.1224$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/\ u = 1/\ 1.1224 = .8909$$

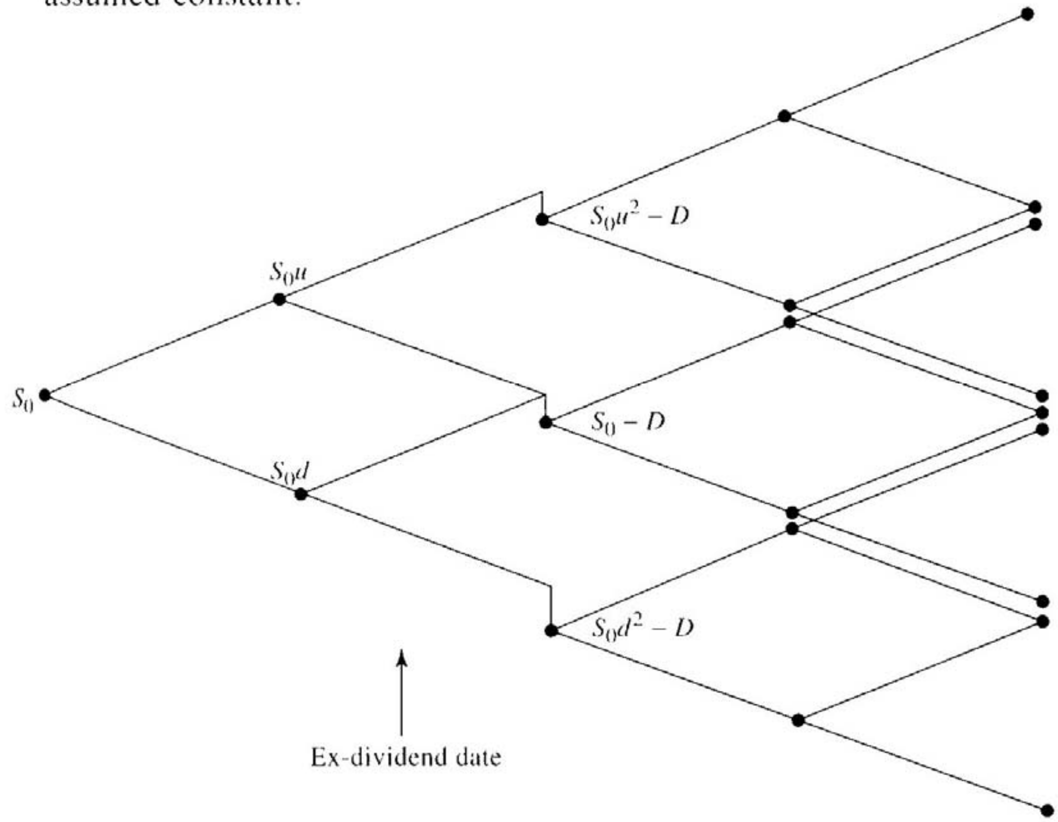
$$p^* = \frac{e^{\mu\Delta t} - d}{u-d} = \frac{e^{.12(\frac{1}{12})} - .8909}{1.1224 - .8909} = .5146$$

American Put

E(R) = 12%
S_0 = $82
σ = 55%
X = $80
R_f = 5%
Expiration = 4 months
Steps = 1 month

Known Dollar Dividend

Figure 19.8 Tree when dollar amount of dividend is assumed known and volatility is assumed constant.



Known Dividend Alternate Procedure

Find and use S^* → $S^* = S_0 - (\text{Dividend})e^{-Rt}$

European Put

E(R) =	12%
S_0 =	$52.00
s =	40%
X =	$50.00
R_f =	5%
D =	$2.06
D due in	3.5 months
Expiration	4 months
Steps	1 month

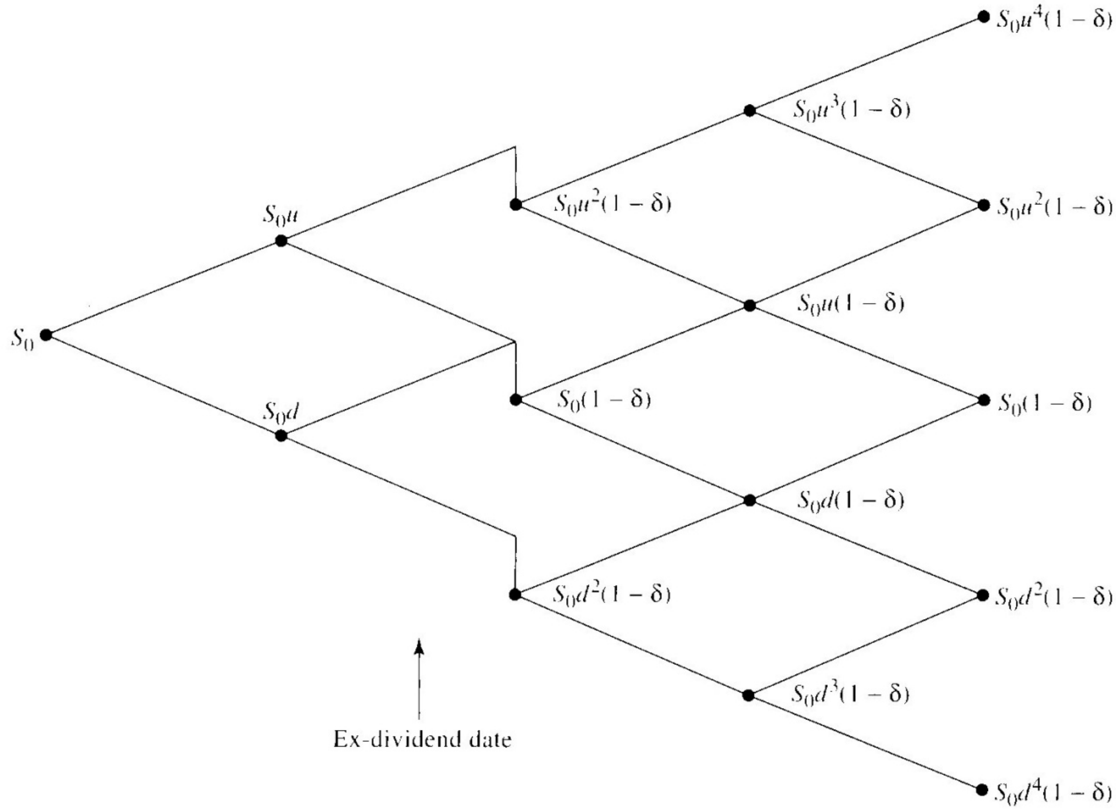
American Call

μ =	13%
S_0 =	$77
σ =	55%
X =	$75
R_f =	5%
D =	$2.35
D due in	3.5
Expiration	5
Steps	1

								S_P*= S_K*u =	165.19
								S_P = S_P*-PV(D) =	165.19
								f_P =	90.19
						S_K* =	140.94		
						S_K =	140.94		
						f_K =	67.20		
				S_G* =	120.25			S_Q* =	120.25
				S_G =	122.60			S_Q =	120.25
				f_G =	47.60			f_Q =	45.25
			S_D*=	102.60			S_L* =	102.60	
			S_D =	104.93			S_L =	102.60	
			f_D =	32.09			f_L =	28.60	
		S_B*	87.53		S_H* =	87.53		S_R* =	87.53
		S_B =	89.86		S_H =	89.88		S_R =	87.53
		f_B =	20.84		f_H =	17.19		f_R =	12.53
S_0*	74.7		S_E* =	74.68			S_M* =	74.68	
S_0 =	77.00		S_E =	77.02			S_M =	74.68	
f_A =	13.12		f_E =	10.00			f_M =	6.17	
		S_C*	63.72		S_I* =	63.72		S_S* =	63.72
		S_C =	66.05		S_I =	66.06		S_S =	63.72
		f_C =	5.68		f_I =	3.04		f_S =	0.00
			S_F* =	54.37			S_N*	54.37	
			S_F =	56.70			S_N =	54.37	
			f_F =	1.50			f_N =	0.00	
				S_J* =	46.38			S_V* =	46.38
				S_J =	48.73			S_V =	46.38
				f_J =	0.00			f_V =	0.00
						S_O* =	39.57		
						S_O =	39.57		
						f_O =	0.00		
								S_W* =	33.76
								S_W =	33.76
								f_W =	0.00

Known Dividend Yield (δ)

Figure 19.7 Tree when stock pays a known dividend yield at one particular time.



Continuous Dividend Yield (δ)

p = $\frac{e^{(R-q)\Delta t} - d}{u-d}$

u = $e^{\sigma\sqrt{\Delta t}}$

d = $e^{-\sigma\sqrt{\Delta t}}$ = 1/ u

6-month European call on the S&P 500, 3 month steps
X = 800 S = 810 R_f = 5% σ = 20% q = 2%

Options on Futures

In a risk neutral world, futures should have a growth rate of zero. So:

$u = e^{\sigma\sqrt{\Delta t}}$

$d = e^{-\sigma\sqrt{\Delta t}} = 1/u$

$p = \frac{1-d}{u-d}$

9 month American put, 3 month steps

$F_0 = 31$ $X = 30$ $\sigma = 30\%$ $R_f = 5\%$