Option Valuation with Black-Scholes Merton

Continuous Compounding

Suppose you invest $100 today for one year at a 10% APR compounded daily. How much will you have in one year?

What is interest is compounded instantaneously (continuously)?

$FV = PVe^{Rt}$

Suppose you invest the $100 for 80 days. How much will you have now?

Suppose you need $1,000 in 5 months and can get a 10% APR compounded continuously. How much do you need to invest today?

$PV = FVe^{-Rt}$

Option Price Boundaries

Can an option ever be worth less than $0?

Suppose a stock is selling for $70 and a call on that stock is selling for $75. What would you do?

Upper bound of a call → Stock price

Suppose a put with a strike price of $70 is selling for $80. What would you do?

Upper bound of a put → Exercise price

<u>Intrinsic value of an option</u> – The payoff an option holder receives is the underlying asset does not change in value.

	Strike	Stock price	Intrinsic value
Call	$50	$45	*$0*
	$50	$55	*$5*
Put	$50	$45	*$5*
	$50	$55	*$0*

Call option intrinsic value = Max[0, S – X]
Put option intrinsic value = Max[0, X – S]

Can an option ever sell for less than intrinsic value? ***NO***

Suppose we have the following:

S = $60
X = $50
C = $4

What is the intrinsic value? Can you make an arbitrage profit?

Therefore, an option can never sell for less than intrinsic value.

Call option price ≥ Max[0, S – X]
Put option price ≥ Max[0, X – S]

An option price has two components, and intrinsic value and a time value.

Suppose we have the following:

S = $73
X = $70
C = $6.43

What is the intrinsic value? What is the time value?

Put-Call Parity

Suppose you do the following:

1) Buy 100 shares of stock.
2) Write one call.
3) Buy one put.

What is your payoff at expiration?

Put-Call Parity with Known Dividend

$C - P = S - (Div)e^{-Rt} - Xe^{-Rt}$

Black-Scholes-Merton Model – Call option pricing

$C_0 = S_0 e^{-yt} N(d_1) - X e^{-Rt} N(d_2)$

$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(R - y + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$

$d_2 = d_1 - \sigma\sqrt{t}$

C_0 = call price today
S_0 = current stock price
X = strike price
R = risk-free rate with the same maturity as the option
y = dividend yield
σ = annual standard deviation of stock return
t = time to maturity
N(x) = cumulative normal probablitity

What is the call price of the following option?

S_0 = \$83	X = \$80	R = 5%
Maturity = 105 days	σ = 45%	y = 2%

Put Pricing (1)

$P = C + Xe^{-Rt} - S_0e^{-yt}$

Black-Scholes-Merton Put Pricing Model

$P_0 = Xe^{-Rt}N(-d_2) - S_0e^{-yt}N(-d_1)$

$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(R - y + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$

$d_2 = d_1 - \sigma\sqrt{t}$

NOTE: $N(-d_1) = 1 - N(d_1)$ Why?

Black-Scholes-Merton is for European options.

How do the inputs affect option prices? For American options:

Input	Sign of Input Effect: Call	Put	Common Name
Underlying stock price (*S*)	+	–	Delta
Strike price of the option contract (*K*)	–	+	
Time remaining until option expiration (*T*)	+	+	Theta
Volatility of the underlying stock price (σ)	+	+	Vega
Risk-free interest rate (*r*)	+	–	Rho
Dividend yield of the underlying stock (*y*)	–	+	

Note: All of the signs are the same for European options except time to maturity which is indeterminate.

The Greeks

Delta – the hedge ratio. Delta measures the dollar impact on an option of a dollar change in the stock price.

Delta of a call = $e^{-yt}N(d_1) > 0$

Delta of a put = $-e^{-yt}N(-d_1) < 0$

$$= -e^{-.02(105/365)}(.378674) = -.37650$$

Eta – Eta measures the percentage change in the option price for a percentage change in the stock price.

Eta of a call = $e^{-yt}N(d_1)(S/C) > 1$

Eta of a put = $-e^{-yt}N(-d_1)(S/P) < 0$

Vega – Vega measures the dollar impact on the option price for a 1% change in the volatility. Vega is the same for a put and a call.

$$\text{Vega} = S_0 e^{-yt} N'(d_1)\sqrt{t} > 0$$

where $N'(x)$ = the normal density function

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-(\frac{x^2}{2})}$$

Gamma – Gamma measures the delta sensitivity. A $1 change in the stock price changes delta by gamma. Gamma is the same for a put and a call.

Gamma = $\frac{N'(d_1)e^{-yt}}{S_0\sigma\sqrt{t}}$

Theta – Theta measures the option price sensitivity to changes in maturity. A 1 day change in maturity causes the option price to change by theta.

Call theta = $-\frac{S_0 N'(d_1)\sigma e^{-yt}}{2\sqrt{t}} + yS_0 N(d_1)e^{-yt} - RXe^{-Rt}N(d_2)$

Put theta = $-\frac{S_0 N'(d_1)\sigma e^{-yt}}{2\sqrt{t}} - yS_0 N(-d_1)e^{-yt} + RXe^{-Rt}N(-d_2)$

Rho – Sensitivity of the option to changes in the interest rate. Rho is the same with and without dividends.

Call rho = $Xte^{-Rt}N(d_2)$

Put rho = $-Xte^{-Rt}N(-d_2)$

Implied Standard Deviation (Implied Volatility)

For a near the money option

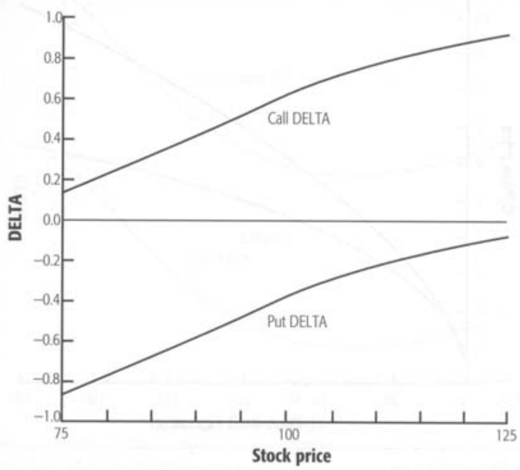
$$\sigma \approx \frac{\sqrt{2\pi/t}}{A+B}\left(C - \frac{A-B}{2} + \sqrt{\left(C - \frac{A-B}{2}\right)^2 - \frac{(A-B)^2}{\pi}}\right)$$

where:

$A = S_0e^{-yt}$
$B = Xe^{-Rt}$

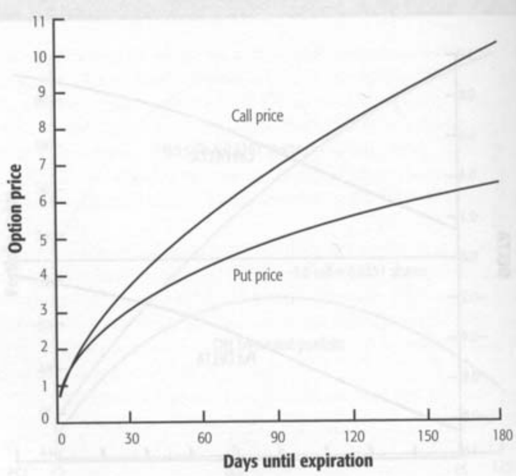
Why is there a volatility smile (skew)?
1) Black-Scholes assumes a constant volatility when in fact there is stochastic volatility.
2) Leverage – As the stock price declines, leverage increases. Equity becomes more risky and volatility increases.
3) "Crashophobia" – Traders are concerned about market crashes and price options accordingly.

Figure 14.4 Call and Put DELTAs as a Function of the Stock Price



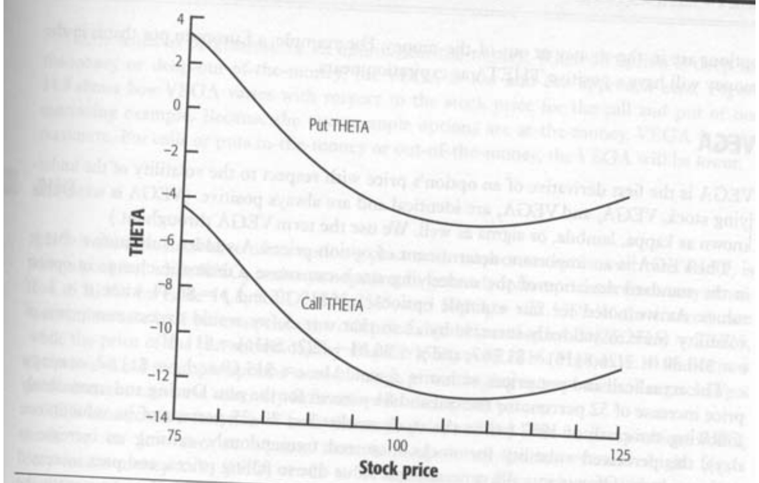
Note: $X = \$100$; $\sigma = .3$; $r = .08$; $T - t = 180$ days.

Figure 14.5 Call and Put Prices as a Function of the Time until Expiration



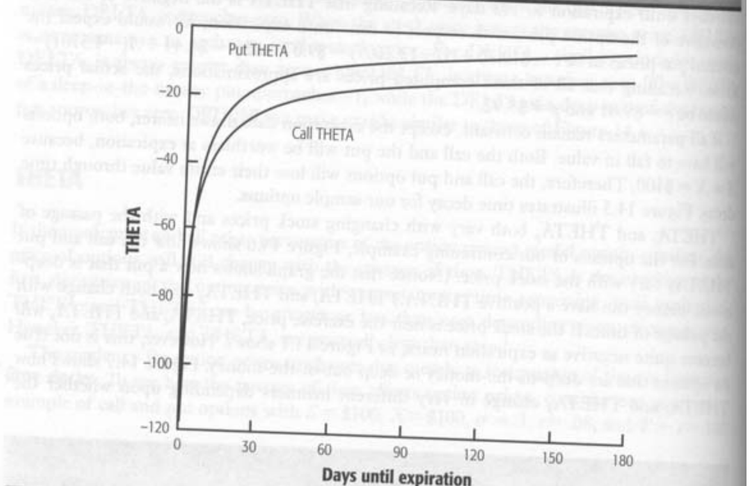
Note: $S = \$100$; $X = \$100$; $\sigma = .3$; $r = .08$.

Figure 14.6 Call and Put THETAs as a Function of the Stock Price



Note: $X = \$100$; $\sigma = .3$; $r = .08$; $T - t = 180$ days.

Figure 14.7 Call and Put THETAs as a Function of the Time until Expiration



Note: $X = \$100$; $\sigma = .3$; $r = .08$.

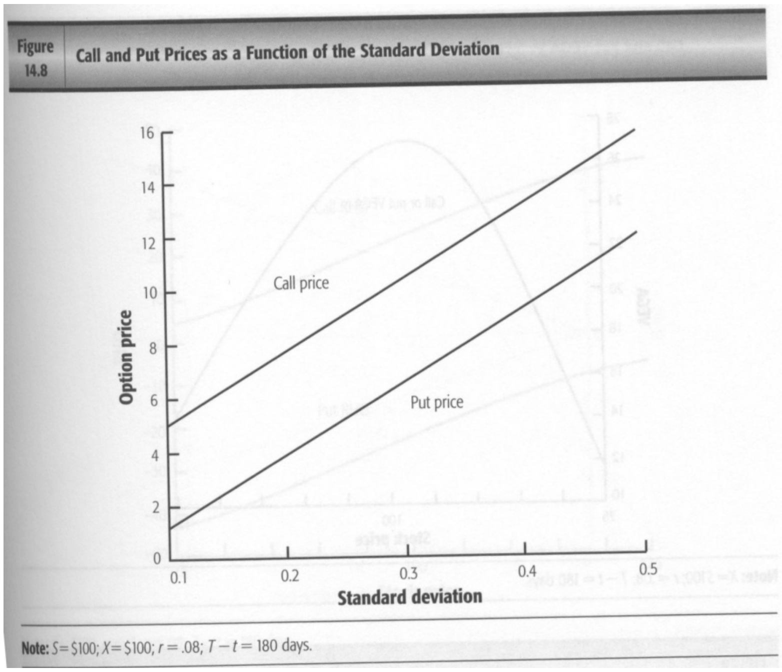


Figure 14.8
Call and Put Prices as a Function of the Standard Deviation
Option price
Call price
Put price
Standard deviation
Note: S = $100; X = $100; r = .08; T − t = 180 days.

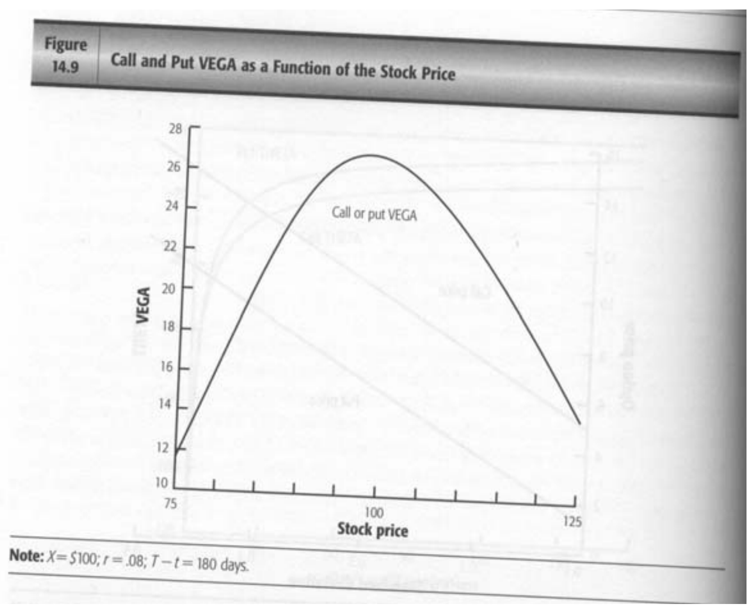


Figure 14.9
Call and Put VEGA as a Function of the Stock Price
VEGA
Call or put VEGA
Stock price
Note: X = $100; r = .08; T − t = 180 days.

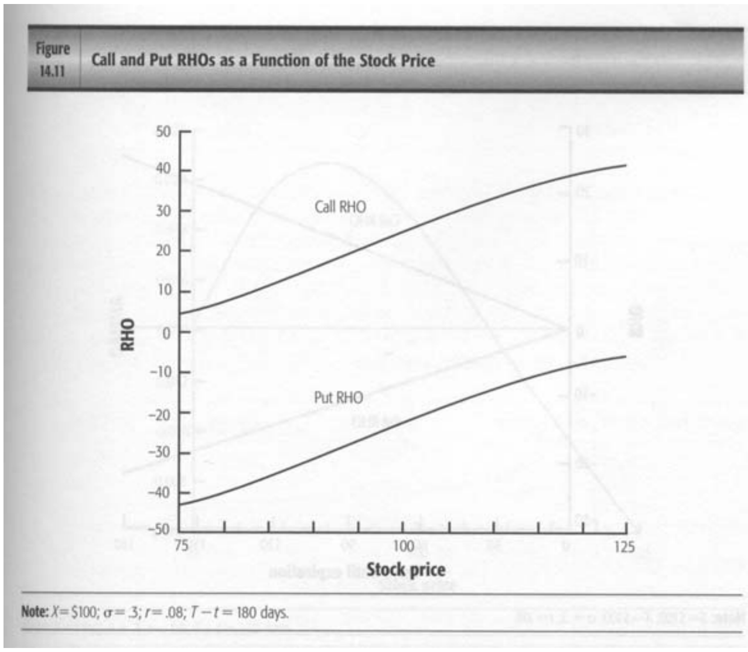


Figure 14.11 Call and Put RHOs as a Function of the Stock Price

Note: $X = \$100$; $\sigma = .3$; $r = .08$; $T - t = 180$ days.

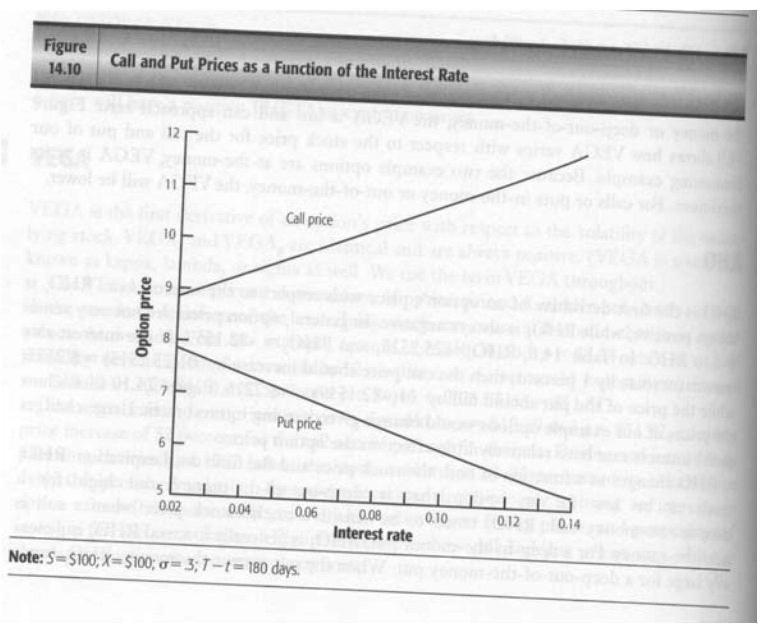
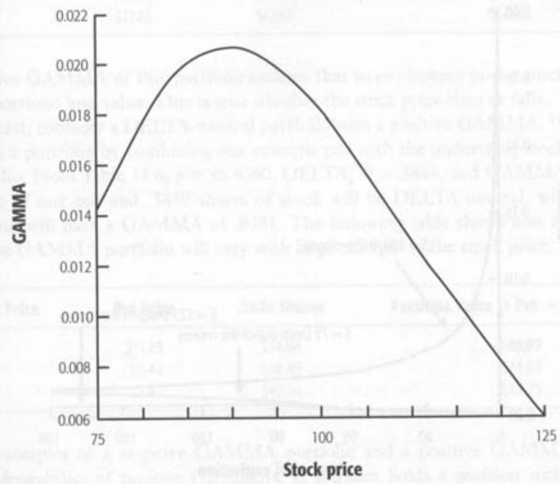


Figure 14.10 Call and Put Prices as a Function of the Interest Rate

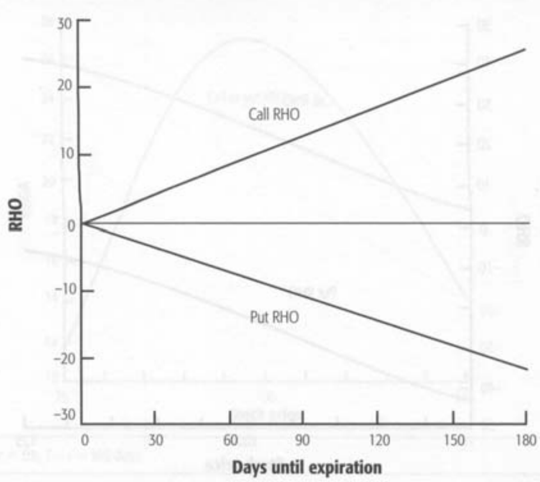
Note: $S = \$100$; $X = \$100$; $\sigma = .3$; $T - t = 180$ days.

Figure 14.13 GAMMA as a Function of the Stock Price



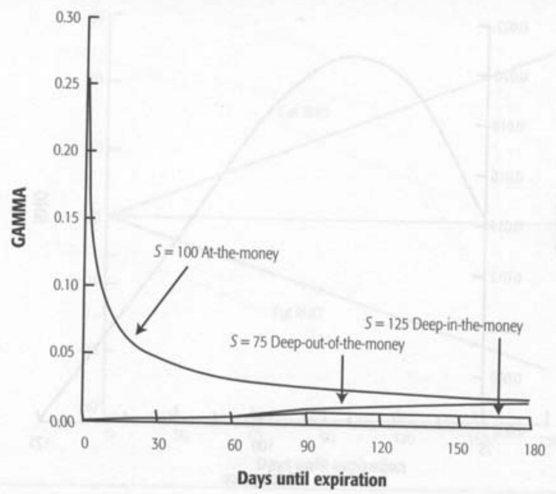
Note: $X = \$100$; $\sigma = .3$; $r = .08$; $T - t = 180$ days.

Figure 14.12 Call and Put RHOs as a Function of the Time until Expiration



Note: $S = \$100$; $X = \$100$; $\sigma = .3$; $r = .08$.

Figure 14.14 **Gamma as a Function of the Time until Expiration**



Note: $X = \$100$; $\sigma = .3$; $r = .08$; $T - t = 180$ days.

Hedging with index options

Number of option contracts = $\frac{\text{Portfolio beta } \times \text{Portfolio value}}{\text{Option delta} \times \text{option contract value}}$

You have a portfolio that has a beta of 1 and a value of $10 million. There are call options on the S&P with a strike price of 1500, 73 days to expiration, and a price of 64.625. The current value of the S&P is 1508.80 and the dividend yield is 1.5%. How many options do you need to hedge your portfolio?

Black-Scholes with trading days

Only a small difference except for very short-life options.

$$t1 = \frac{\text{trading days until maturity}}{\text{trading days per year}}$$

$$t2 = \frac{\text{calendar days until maturity}}{\text{calendar days per year}}$$

$$C_0 = S_0N(d_1) - Xe^{-R(t2)}N(d_2)$$

$$P_0 = Xe^{-R(t2)}N(-d_2) - S_0e^{-yt}N(-d_1)$$

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(R(t2) + \frac{\sigma^2(t1)}{2}\right)}{\sigma\sqrt{(t1)}}$$

$$d_2 = d_1 - \sigma\sqrt{t1}$$