Duration

1) Average life of an asset, or the weighted average time to maturity

2) Measure of the elasticity of price in relation to interest rates

$$\frac{\partial P}{P} = -D\left(\frac{\partial R}{1+R}\right)$$

D = 4 years, ∂R = 50 basis point increase, R = YTM = 10%

With semi-annual coupons

$$\frac{\partial P}{P} = -D\left(\frac{\partial R}{1+(R/2)}\right)$$

Calculating Duration
8% annual coupon bond, 4 years to maturity, YTM = 10%

Method 1 – discount at YTM

t	CF	DCF	DCF × t
1	80	72.73	72.73
2	80	66.12	132.24
3	80	60.11	180.33
4	1,080	737.65	2950.60
		Price = 936.61	3,335.90

$$D = \frac{\sum DCF \times t}{\sum DCF\,(price)} = \frac{3{,}335.90}{936.61} = 3.562 \text{ years}$$

Method 2 – discount at YTM

t	CF	DCF	Weight	t* weight
1	80	72.73	72.73 / 936.61 = .07765	.07765
2	80	66.12	66.12 / 936.61 = .07060	.14119
3	80	60.11	60.11 / 936.61 = .06418	.19253
4	1,080	737.65	737.65 / 936.61 = .78757	3.15029
		Price = 936.61		3.562 years

Method 3 (Formula)

$$D = \frac{1+y}{y} - \frac{(1+y) + T(c - y)}{c[(1+y)^T - 1] + y}$$

For a semi-annual bond
2 year, 8% semi-annual coupon, 10% YTM

t	CF	DCF	DCF × t
1	40	38.10	19.05
2	40	36.28	36.28
3	40	34.55	51.83
4	1,040	855.61	1,711.22
		Price = 964.54	1,818.38

$$D = \frac{\sum DCF \times t}{\sum DCF\,(\text{price})} = \frac{1{,}818.38}{964.54} = 1.885 \text{ years}$$

15 years, 10% semi-annual coupon, 9% YTM

Value of Duration

Reinvestment Risk – risk of being unable to reinvest payments at the same YTM. For the holding period return to equal the YTM, all coupons must be reinvested at the YTM.

Price risk (interest rate risk) – purchase a bond with a longer term to maturity than assets, i.e. purchase an 8-year bond to fulfill your 5-year liability.

Example – You owe $100 million in five years. The YTM is 8%. How much do you have to purchase in par value of 8% coupon bonds to meet the liability?

Alternative:

Purchase bonds with a duration of 5 years. No matter what the reinvestment rate (assuming it is constant, i.e. only a one-time interest rate change), the terminal value in 5 years will be $100M. The reason: The price risk and reinvestment risk offset.

Duration concepts

1) Holding maturity constant, a bond's duration is higher when the coupon rate is lower.

2) Duration increases as maturity increases. The increase in duration is at a decreasing rate. (This always applies to par or premium bonds, however some deep discount bonds can decrease duration as maturity increases, but this is rare.)

3) Duration for coupon bonds is higher when YTM is lower.

4) Duration for a zero coupon is maturity.

5) Duration of a perpetuity is: $\frac{1+y}{y}$

6) Duration for a level annuity is: $\frac{1+y}{y} - \frac{T}{(1+y)^T - 1}$

Using Duration for price changes
10 year, 10% annual coupon, 10% YTM

$$D = \frac{1+y}{y} - \frac{(1+y)+T(c-y)}{c[(1+y)^T - 1] + y} = \frac{1+.10}{.10} - \frac{(1+.10)+10(.10-.10)}{.10[(1+.10)^{10} - 1]+.10} = 6.759 \text{ yrs}$$

Duration model is less accurate when interest rates change more

YTM increases by 2% to 12%

Calculated price = $886.99

$$\frac{\partial P}{P} = -D\left[\frac{\partial R}{1+R}\right] = -6.759\left[\frac{.02}{1+.01}\right] = -12.289\%$$

$1,000 (1 + (-12.289) = $877.11

For a semi-annual bond

$$\frac{\partial P}{P} = -D\left[\frac{\partial R}{1+\frac{1}{2}R}\right]$$

To calculate a dollar increase (decrease)

$$\partial P = P \times [(-D) \times \left[\frac{\partial R}{1+R}\right]]$$

Immunization

Problems with Immunization using Duration

1) Duration matching can be costly.

2) Duration is dynamic.

3) Large interest rate changes

Convexity Adjustment

$$\frac{\partial P}{P} = -D\left[\frac{\Delta R}{1+R}\right] + \frac{1}{2}CX(\Delta R)^2$$

CX = convexity = Scaling factor[capital loss from one basis point rise in R + capital gain from one basis point drop in R]

scaling factor = 10^8

Convexity rules

1) Convexity increases with bond maturity

2) Convexity varies with coupon rate

3) For the same duration, a zero coupon is less convex than a coupon bond

4) Duration and Convexity of a portfolio are the weighted average of the portfolio assets

** Seek greater convexity in asset portfolio

Modified Duration

$D_M = \frac{D}{1+y}$

%Δ in bond price = $-D_M(\Delta R)$

8% semiannual coupon, 20 year maturity, 10% YTM
Macaulay duration = 7.77449 years

$$D_M = \left[\frac{7.77449}{1+.10/2}\right]$$

D_M = 7.0938 years

If interest rates rise 25 basis points:

%Δ in bond price = –7.0938(.0025) = –.0177 or 1.77%

Effective (Approximate) Duration

Advantages:
1) Doesn't use the YTM, which may be unimportant.
2) Can account for embedded options.

$$D_E = \frac{V_- - V_+}{2V_0(\Delta R)}$$

V_0 = initial price
V_- = price if YTM decreases by R
V_+ = price if YTM increases by R

18 year, 7% semiannual coupon, 8% YTM, change in YTM = 20 basis points

V_0 = \$905.46
V_- = \$923.31
V_+ = \$888.10

Effective Convexity

$$CX_E = \frac{V_- + V_+ - 2V_0}{2V_0(\Delta R)^2}$$

Problems with Effective Convexity

Not measured the same

$$CX_E = \frac{V_{-} + V_{+} - 2V_0}{V_0(\Delta R)^2}$$

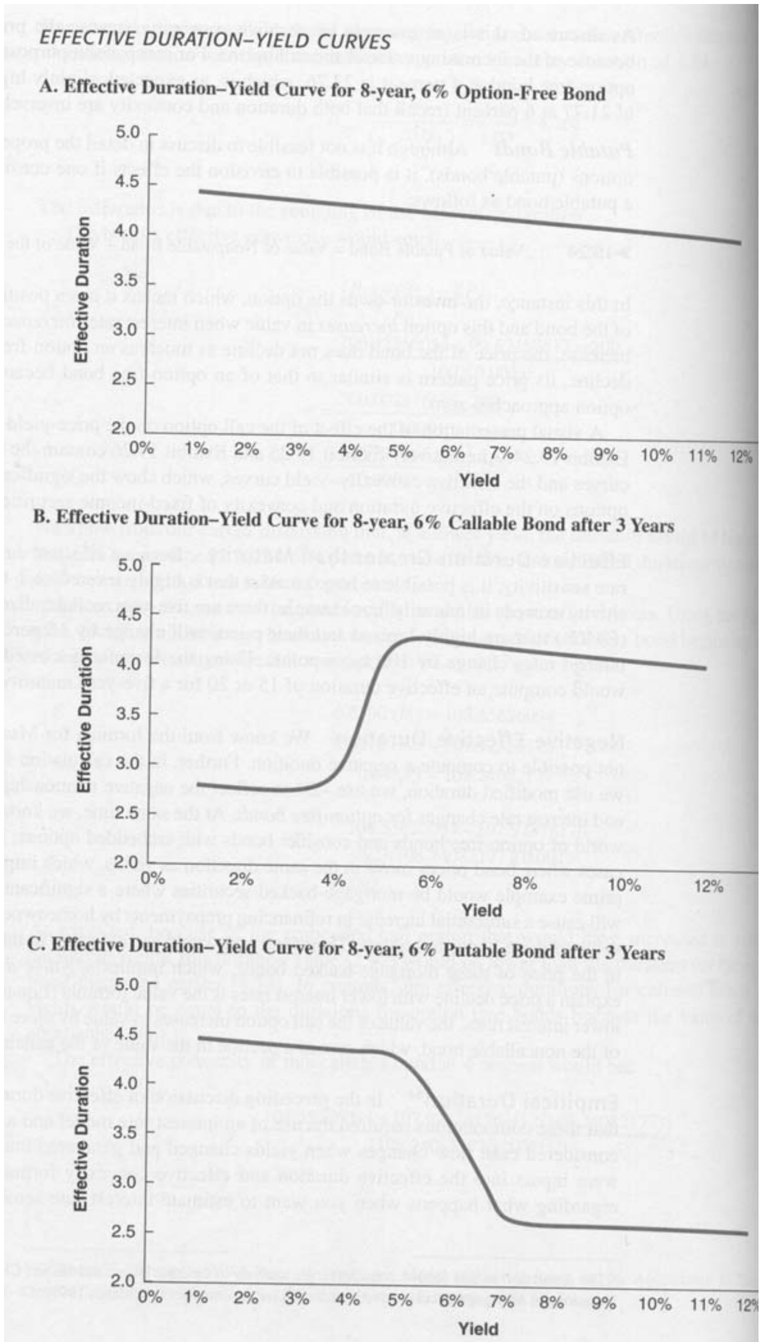
NOTE: 2 in denominator is deleted

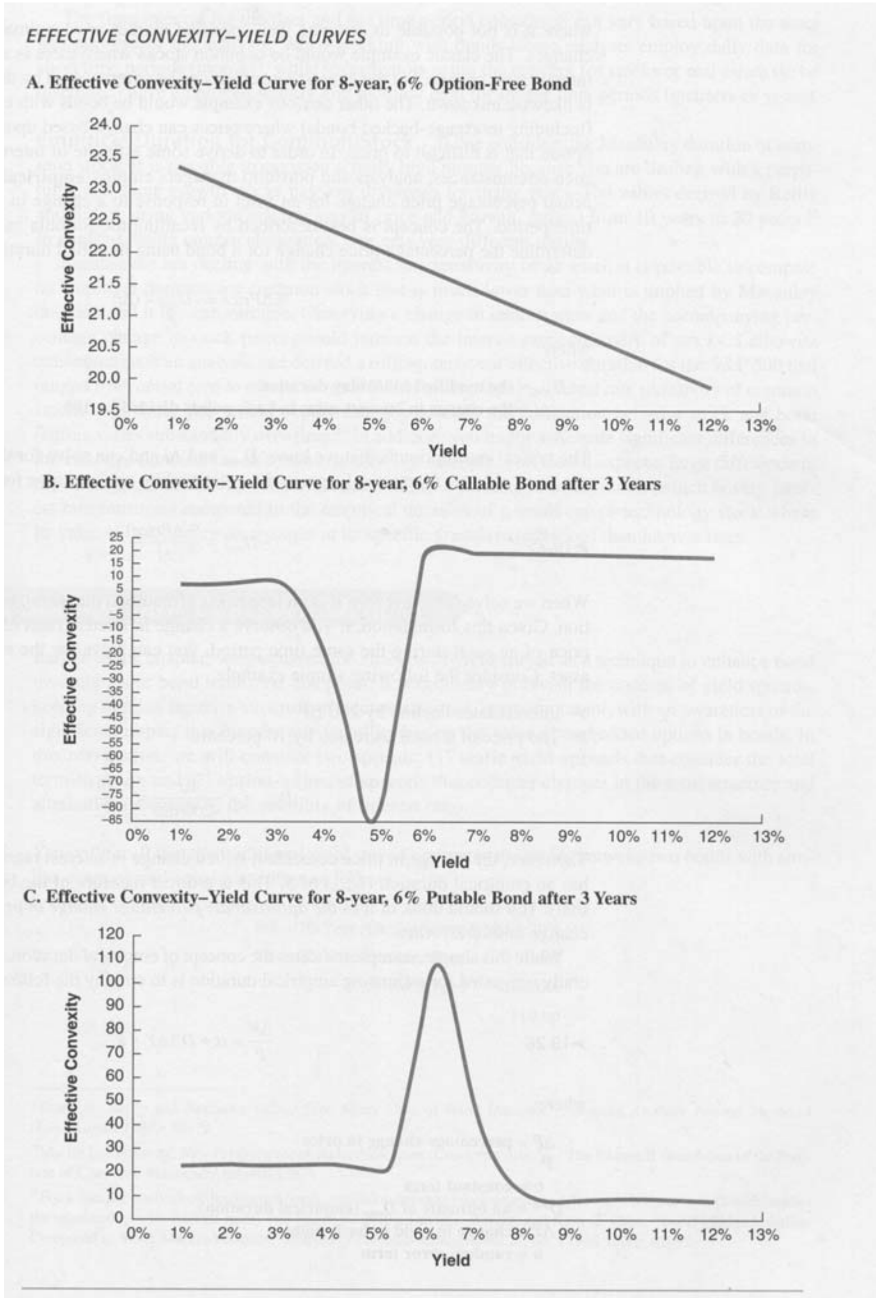
Convexity effect = $(½)CX_E(\Delta R)^2$

Why is duration important?

1) To immunize a portfolio, match the duration and not the maturity

2) Duration (and convexity) are weighted averages of the duration (convexity) of the assets in the portfolio.


EFFECTIVE DURATION–YIELD CURVES
A. Effective Duration–Yield Curve for 8-year, 6% Option-Free Bond
Effective Duration
5.0
4.5
4.0
3.5
3.0
2.5
2.0
0% 1% 2% 3% 4% 5% 6% 7% 8% 9% 10% 11% 12%
Yield
B. Effective Duration–Yield Curve for 8-year, 6% Callable Bond after 3 Years
Effective Duration
5.0
4.5
4.0
3.5
3.0
2.5
2.0
0% 2% 4% 6% 8% 10% 12%
Yield
C. Effective Duration–Yield Curve for 8-year, 6% Putable Bond after 3 Years
Effective Duration
5.0
4.5
4.0
3.5
3.0
2.5
2.0
0% 1% 2% 3% 4% 5% 6% 7% 8% 9% 10% 11% 12%
Yield



EFFECTIVE CONVEXITY–YIELD CURVES

A. Effective Convexity–Yield Curve for 8-year, 6% Option-Free Bond

B. Effective Convexity–Yield Curve for 8-year, 6% Callable Bond after 3 Years

C. Effective Convexity–Yield Curve for 8-year, 6% Putable Bond after 3 Years

Yield value of a 32nd

Treasury bond selling at par with a 10% coupon (semiannual)

Coupon	Maturity	Initial price minus 1/32	New YTM	Initial Yield	Yield value of a 32nd
10%	5	99.96875	10.008096%	10%	0.008096%
10%	15	99.96875	10.004067%	10%	0.004067%
10%	30	99.96875	10.003303%	10%	0.003303%

Yield value of an 8th – Muni bonds

Municipal bond selling at par with a 10% coupon (semiannual)

Coupon	Maturity	Initial price minus 1/8	New YTM	Initial Yield	Yield value of an 8th
10%	5	99.875	10.032402%	10%	0.032402%
10%	15	99.875	10.016278%	10%	0.016278%
10%	30	99.875	10.013222%	10%	0.013222%

Dollar value of a basis point (Value of an ’01)

Coupon	Maturity	Initial YTM 10.00%	YTM changes 1 bp 10.01%	Value of an '01
8%	5	$922.7827	$922.4154	$0.3672
8%	15	$846.2755	$845.5949	$0.6806
8%	30	$810.7071	$809.9200	$0.7871
14%	5	$1,154.4347	$1,154.0112	$0.4235
14%	15	$1,307.4490	$1,306.5057	$0.9433
14%	30	$1,378.5858	$1,377.3229	$1.2628

Dollar value of an ’01 is higher:
1) For higher coupons.
2) For lower yields.