Option Valuation with Black-Scholes Merton

Continuous Compounding

Suppose you invest $100 today for one year at a 10% APR compounded daily. How much will you have in one year?

What is interest is compounded instantaneously (continuously)?

\[ FV = PV e^{Rt} \]

Suppose you invest the $100 for 80 days. How much will you have now?

Suppose you need $1,000 in 5 months and can get a 10% APR compounded continuously. How much do you need to invest today?

\[ PV = FVe^{-Rt} \]
Option Price Boundaries

Can an option ever be worth less than $0?

Suppose a stock is selling for $70 and a call on that stock is selling for $75. What would you do?

Upper bound of a call $\rightarrow$ Stock price

Suppose a put with a strike price of $70 is selling for $80. What would you do?

Upper bound of a put $\rightarrow$ Exercise price

Intrinsic value of an option – The payoff an option holder receives is the underlying asset does not change in value.

<table>
<thead>
<tr>
<th></th>
<th>Strike</th>
<th>Stock price</th>
<th>Intrinsic value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>$50</td>
<td>$45</td>
<td>$0</td>
</tr>
<tr>
<td></td>
<td>$50</td>
<td>$55</td>
<td>$5</td>
</tr>
<tr>
<td>Put</td>
<td>$50</td>
<td>$45</td>
<td>$5</td>
</tr>
<tr>
<td></td>
<td>$50</td>
<td>$55</td>
<td>$0</td>
</tr>
</tbody>
</table>

Call option intrinsic value = Max[0, S – X]
Put option intrinsic value = Max[0, X – S]

Can an option ever sell for less than intrinsic value? NO
Suppose we have the following:
\[ S = 60 \]
\[ X = 50 \]
\[ C = 4 \]
What is the intrinsic value? Can you make an arbitrage profit?

Therefore, an option can never sell for less than intrinsic value.

\[
\text{Call option price} \geq \text{Max}[0, S - X] \\
\text{Put option price} \geq \text{Max}[0, X - S]
\]

An option price has two components, and intrinsic value and a time value.

Suppose we have the following:
\[ S = 73 \]
\[ X = 70 \]
\[ C = 6.43 \]
What is the intrinsic value? What is the time value?
Put-Call Parity

Suppose you do the following:
1) Buy 100 shares of stock.
2) Write one call.
3) Buy one put.
What is your payoff at expiration?

Put-Call Parity with Known Dividend
\[ C - P = S - (\text{Div})e^{-Rt} - Xe^{-Rt} \]
Black-Scholes-Merton Model – Call option pricing

\[ C_0 = S_0 e^{-y t} N(d_1) - X e^{-R t} N(d_2) \]

\[ d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( R - y + \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}} \]

\[ d_2 = d_1 - \sigma \sqrt{t} \]

\( C_0 \) = call price today
\( S_0 \) = current stock price
\( X \) = strike price
\( R \) = risk-free rate with the same maturity as the option
\( y \) = dividend yield
\( \sigma \) = annual standard deviation of stock return
\( t \) = time to maturity
\( N(x) \) = cumulative normal probability

What is the call price of the following option?

\( S_0 = \$83 \quad X = \$80 \quad R = 5\% \)
\( \text{Maturity} = 105 \text{ days} \quad \sigma = 45\% \quad y = 2\% \)
Put Pricing (1)

\[ P = C + X e^{-Rt} - S_0 e^{-yt} \]

Black-Scholes-Merton Put Pricing Model

\[ P_0 = X e^{-Rt} N(-d_2) - S_0 e^{-yt} N(-d_1) \]

\[ d_1 = \frac{\ln \left( \frac{X}{S_0} \right) + (R - y + \frac{\sigma^2}{2})t}{\sigma \sqrt{t}} \]

\[ d_2 = d_1 - \sigma \sqrt{t} \]

NOTE: \( N(-d_1) = 1 - N(d_1) \)  Why?
Black-Scholes-Merton is for European options.

How do the inputs affect option prices? For American options:

<table>
<thead>
<tr>
<th>Input</th>
<th>Sign of Input Effect</th>
<th>Common Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underlying stock price ($S$)</td>
<td>+</td>
<td>Delta</td>
</tr>
<tr>
<td>Strike price of the option contract ($K$)</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td>Time remaining until option expiration ($T$)</td>
<td>$+$</td>
<td>Theta</td>
</tr>
<tr>
<td>Volatility of the underlying stock price ($\sigma$)</td>
<td>$+$</td>
<td>Vega</td>
</tr>
<tr>
<td>Risk-free interest rate ($r$)</td>
<td>$+$</td>
<td>Rho</td>
</tr>
<tr>
<td>Dividend yield of the underlying stock ($y$)</td>
<td>$-$</td>
<td></td>
</tr>
</tbody>
</table>

Note: All of the signs are the same for European options except time to maturity which is indeterminate.
The Greeks

**Delta** – the hedge ratio. Delta measures the dollar impact on an option of a dollar change in the stock price.

Delta of a call = \( e^{-\gamma t}N(d_1) > 0 \)

\[
= e^{-0.02(105/365)} \cdot 0.37674 = -0.37650
\]

Delta of a put = \(-e^{-\gamma t}N(-d_1) < 0\)
**Eta** – Eta measures the percentage change in the option price for a percentage change in the stock price.

Eta of a call = $e^{-yt}N(d_1)(S/C) > 1$

Eta of a put = $-e^{-yt}N(-d_1)(S/P) < 0$
**Vega** – Vega measures the dollar impact on the option price for a 1% change in the volatility. Vega is the same for a put and a call.

\[
\text{Vega} = S_0 e^{-\gamma t} N'(d_1) \sqrt{t} > 0
\]

where \( N'(x) = \) the normal density function

\[
N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)}
\]
**Gamma** – Gamma measures the delta sensitivity. A $1 change in the stock price changes delta by gamma. Gamma is the same for a put and a call.

\[
\text{Gamma} = \frac{N'(d_1)e^{-yt}}{S_0\sigma\sqrt{t}}
\]
**Theta** – Theta measures the option price sensitivity to changes in maturity. A 1 day change in maturity causes the option price to change by theta.

Call theta = \[-\frac{S_0N'(d_1)\sigma e^{-yt}}{2\sqrt{t}} + yS_0N(d_1)e^{-yt} - RXe^{-Rt}N(d_2)\]

Put theta = \[-\frac{S_0N'(d_1)\sigma e^{-yt}}{2\sqrt{t}} - yS_0N(-d_1)e^{-yt} + RXe^{-Rt}N(-d_2)\]
**Rho** – Sensitivity of the option to changes in the interest rate. Rho is the same with and without dividends.

Call rho = $Xte^{-Rt}N(d_2)$

Put rho = $-Xte^{-Rt}N(-d_2)$
**Implied Standard Deviation (Implied Volatility)**

For a near the money option

\[\sigma \approx \sqrt{\frac{2\pi / \nu}{A+B}} \left(C - \frac{A-B}{2} + \sqrt{\left(C - \frac{A-B}{2}\right)^2 - \frac{(A-B)^2}{\pi}}\right)\]

where:

\[A = S_0e^{-yt}\]

\[B = Xe^{-Rt}\]

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Why is there a volatility smile (skew)?
1) Black-Scholes assumes a constant volatility when in fact there is stochastic volatility.
2) Leverage – As the stock price declines, leverage increases. Equity becomes more risky and volatility increases.
3) “Crashophobia” – Traders are concerned about market crashes and price options accordingly.
Figure 14.4  Call and Put DELTAs as a Function of the Stock Price

Note: $X = \$100; \sigma = 3; r = .06; T - t = 100$ days.

Figure 14.5  Call and Put Prices as a Function of the Time until Expiration

Note: $S = \$100; X = \$100; \sigma = 3; r = .08$.
Figure 14.6 Call and Put THETA as a Function of the Stock Price

Note: $X = 500, \sigma = 3, r = 0.08, T - t = 180$ days.

Figure 14.7 Call and Put THETA as a Function of the Time until Expiration

Note: $X = 500, \sigma = 3, r = 0.08$. 
Figure 14.8  Call and Put Prices as a Function of the Standard Deviation

Call price
Put price

Note: S = $100, X = $100, r = 0.06, T - t = 180 days.

Figure 14.9  Call and Put VEGA as a Function of the Stock Price

Call or put VEGA

Note: X = $100; r = 0.06; T - t = 180 days.
Figure 14.33: GAMMA as a Function of the Stock Price

Note: \( X = 5100; \sigma = 3; r = 0.08; T - t = 180 \) days.

Figure 14.32: Call and Put RHOs as a Function of the Time until Expiration

Note: \( S = 51000; X = 51000; \sigma = 0.5; r = 0.08 \).
Figure 14.14: Gamma as a Function of the Time until Expiration

Note: $X = 500$, $\sigma = 0.5$, $r = 0.05$, $T - t = 180$ days.
Hedging with index options

Number of option contracts = \( \frac{\text{Portfolio beta} \times \text{Portfolio value}}{\text{Option delta} \times \text{option contract value}} \)

You have a portfolio that has a beta of 1 and a value of $10 million. There are call options on the S&P with a strike price of 1500, 73 days to expiration, and a price of 64.625. The current value of the S&P is 1508.80 and the dividend yield is 1.5%. How many options do you need to hedge your portfolio?
**Black-Scholes with trading days**

Only a small difference except for very short-life options.

\[ t_1 = \frac{\text{trading days until maturity}}{\text{trading days per year}} \]

\[ t_2 = \frac{\text{calendar days until maturity}}{\text{calendar days per year}} \]

\[ C_0 = S_0 N(d_1) - X e^{-R(t_2)} N(d_2) \]

\[ P_0 = X e^{-R(t_2)} N(-d_2) - S_0 e^{-y t} N(-d_1) \]

\[ d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( R(t_2) + \frac{\sigma^2(t_1)}{2} \right)}{\sigma \sqrt{t_1}} \]

\[ d_2 = d_1 - \sigma \sqrt{t_1} \]